The rth column of the $p \times (n-m)$ matrix $\hat{J}$ is given by

$$\frac{\partial X}{\partial q_r} + \sum_{s=1}^{m} C_{sr} \frac{\partial X}{\partial q_{n-m+s}}, \quad r = 1,2,\ldots,n-m$$

and

$$\hat{b} = \sum_{s=1}^{m} \frac{\partial X}{\partial q_{n-m+s}} d_s + \frac{\partial X}{\partial t}$$

We must proceed with caution in constructing a basis for the tangent space. Returning to the example given in the previous section, we saw that when q_1 and q_2 vary *independently*, the particles admissible path is a *surface*. If the variations in q_1 and q_2 are *dependent*, then the admissible path is a *curve* in the surface. The dimension of the tangent space is reduced from two to one. In the current situation, the constraints [Eq. (6)] reduce the number of independent variations of generalized coordinates from n to $n-m$.

In terms of variations of independent generalized coordinates, we have [cf. Eq. (7)]

$$\delta \boldsymbol{x} = \hat{J} \delta \boldsymbol{q}^{(1)}$$

where $\delta \boldsymbol{x}$ are the virtual displacements of the particles satisfying the instantaneous equations of constraint. The vector $\delta \boldsymbol{x}$ lies in the tangent space to the hypersurface. Again we see that the columns of the matrix appearing in the velocity transformation form a basis for the tangent space. Proceeding exactly as in the holonomic case, we can project the equations of motion onto the tangent space. Partitioning these equations by individual particles we obtain

$$\sum_{i=1}^{N} m_i \hat{J}_i^T \hat{J}_i \ddot{\boldsymbol{q}}^{(1)} + \sum_{i=1}^{N} m_i \hat{J}_i^T \dot{\hat{J}}_i \dot{\boldsymbol{q}}^{(1)} + \sum_{i=1}^{N} m_i \hat{J}_i^T \dot{\hat{\boldsymbol{b}}}^{(i)} = \sum_{i=1}^{N} \hat{J}_i^T \boldsymbol{f}^{(i)} \tag{8}$$

Kane's equations for a nonholonomic system are given by Eq. (5), with $r = 1,2,\ldots,n-m$. The partial velocity $V_r^{(i)}$ is the coefficient of $\dot{q}_r$ in the expression for the velocity of particle "i" in terms of the independent generalized velocities. Resolving all vectors in Eq. (5) into the inertial frame and observing that $V_r^{(i)}$ is simply the rth column of $\hat{J}_i$, we see that Kane's equations become identical to Eq. (8)—obtained by the projection method.

Summary

A derivation of Kane's equations based on orthogonal projections is presented. Although less succinct than the traditional approach, it offers some insight into the physics embodied in the equations and appears as a natural generalization of methods used in elementary problems. It is hoped that those familiar with Kane's equations will find the current derivation illuminating and that instructors will adopt it into the classroom setting.

Acknowledgments

We wish to thank Donald Keene of the Draper Laboratory for bringing Ref. 4 to our attention, and Professor Andreas Von Flowtow (of MIT) for stimulating discussions on this topic.

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Improved Time-Domain Stability Robustness Measures for Linear Regulators

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Introduction

THE robustness of multivariable control systems, i.e., their ability to maintain performance in the face of uncertainties and perturbations, has been extensively studied in the past. The recently published literature on robustness analysis of linear time-invariant systems can be viewed from three perspectives: 1) the frequency domain approach, 2) the time domain approach, and 3) the frequency domain approach, which uses a state space representation of the system.

Although many of the robustness criteria developed so far are in the frequency domain,[1-5] it is also useful to analyze stability robustness of multivariable control systems in the time domain, especially when a broader class of parameter perturbations have to be considered, which appears in the state equations describing the plant. The time domain approach[6-12] is primarily based on the Lyapunov theory and generally involves checking only a finite number of inequalities, often just one, while the frequency domain methodology requires that all criteria over the whole range of frequencies be satisfied.

Recently, new bounds on linear time-invariant perturbations that do not destabilize the system were given, based on the frequency domain approach that uses a state space representation of the system.[13-14] It was shown that these bounds are superior to time-domain stability robustness criteria[6,11,12] in the sense that they are less conservative and that they can be applied to a more general class of systems and perturbations.

In this Note, we present a new time domain stability criterion for linear state space models. A computationally effective algorithm is proposed, which leads to perturbation bounds that are superior to those based on the frequency domain approach[13,14] and the time-domain approach.[6,11,12]

Problem Formulation and Development

Consider a linear time-invariant model of a physical system with linear time-invariant perturbations

$$\dot{x}(t) = (A + E)x(t) \tag{1}$$

where $\boldsymbol{x}$ is the n-dimensional state vector, A is an $n \times n$ asymptotically stable matrix, and E is a perturbation matrix. Two types of perturbations have been considered: unstructured perturbations and structured perturbations.

In what follows, a brief discussion on some common approaches to the stability robustness analysis of state space models is given. We restrict our attention only to structured perturbations.

Theorem 1[12]: Assume that the elements of the perturbation matrix E, Eq. (1), are restricted so that

$$|E_{ij}| \le e_{ij} \tag{2}$$

and let $e = \max_{i,j} e_{ij}$. Then the system (1) is stable if

$$e < \frac{1}{\sigma_{\max}[(|P|U)_s]} = \mu_y \tag{3}$$

Received Nov. 16, 1987; revision received April 21, 1988.

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where P is the solution of the Lyapunov matrix equation

$$A^TP + PA + 2I = 0 \tag{4}$$

U is a matrix with elements $U_{ij} = (e_{ij}/e)$, $\sigma_{\max}(\cdot)$ is the maximum singular value of $(\cdot)$, $|(\cdot)|$ denotes the modulus matrix, and $(\cdot)_s$ denotes a symmetric part of the matrix $(\cdot)$.

***Theorem 2*[13]: Assume that the perturbation matrix E, Eq. (1), has the structure**

$$E = S_1E^*S_2 \tag{5}$$

where $S_1 \epsilon R^{n\times p}$, $E^* \epsilon R^{p\times q}$, $S_2 \epsilon R^{q\times n}$, $p \le n$, $q \le n$, and S_1, S_2 are known matrices. With no loss of generality, assume that rank $S = p$ or rank $S_2 = q$, and let the elements of the perturbation matrix be denoted by E^*_{ij}, and assume that

$$|E^*_{ij}| \le e_{ij}\, e \tag{6}$$

where $e_{ij} \le 0$ are given, and $e > 0$ is unknown. Then the perturbed system (1) is stable if

$$e < \frac{1}{\sup_{w\ge 0} \pi(|S_2(j\omega - A)^{-1}S_1|U)} = \mu_Q \tag{7}$$

where $U \epsilon R^{q\times p}$ is a matrix with elements given by $u_{ij} = e_{ij}$, and π denotes perron eigenvalue of a nonnegative square matrix.

It should be pointed out that the frequency domain criterion [Eq. (7)], as well as the time domain criterion [Eq. (3)], in many cases are not adequate in treating structured perturbations, since the directional information cannot be properly represented nor adequately studied. Namely, the perturbation characterization given by Eqs. (3) and (5), i.e., Eq. (6) leads to a loss of directional information. Therefore, although the effect of the plant parameter variations and modeling errors may be incorporated into the structured uncertainties, the directional information associated with the structured uncertainties is lost, and the result is an overly conservative stability robustness bound.

In the next section, we circumvent these difficulties by introducing a new iterative algorithm for stability robustness bounds improvement. In addition, directional information (in the state space model) associated with the structured uncertainty is exploited to reduce conservatism.

An Algorithm for Stability Robustness Bounds Improvement

In what follows, we give a new stability robustness test for improved perturbation bounds. These bounds are superior to those reported in Refs. 11-13 in two senses: 1) they are less conservative, and 2) the directional information on structured perturbations can be properly represented and adequately incorporated in the robustness test.

Note that the scalars e_{ij}, Eqs. (2) and (6), are restricted to be positive. In other words, the directional information on structured perturbations cannot be properly represented by Eqs. (3) and (7). However, in many practical situations a designer following his intuition and experience has enough information concerning the nature of the perturbations (modeling uncertainties, modeling reductions, or parameter variations), which can physically occur, to select the most appropriate directions in the space of all perturbation matrices.

To overcome this difficulty, we will suppose that the perturbation matrix E is defined by

$$E = e\bar{E} \tag{8}$$

where $\bar{E}$ is given, and $e > 0$ is unknown. In what follows, a time-domain stability robustness methodology will be used to develop an iterative algorithm for determining computationally the largest positive number e such that the perturbed system (S_O):

$$\dot{x}(t) = (A + e\bar{E})x(t) \tag{9}$$

where A is a time-invariant asymptotically stable matrix, $\bar{E}$ is given, and $e > 0$ is unknown, remains asymptotically stable.

First we give the following theorem:

Theorem 3: The system (S_O) [Eq. (9)], is stable if

$$e < \frac{1}{\sigma_{\max}[(|P||\bar{E}|)_s]} \tag{10}$$

where P is the solution of the Lyapunov matrix [Eq. (4)], and $\bar{E}$ is a given perturbation matrix,

Proof: The proof follows directly from the proof of Theorem 1,[11] and is omitted.

Now we review the procedure for stability robustness bounds improvement.

Step 1: Using the criterion of condition (10), determine e_0.

Step 2: Consider the perturbed system (S_i) as an unperturbed system, with

$$A = A + \sum_{p=0}^{i-1} e_p\bar{E} \tag{11}$$

and determine e_i, solving the corresponding Lyapunov equation

$$(A + \sum_{p=0}^{i-1} e_p\bar{E})^TP + P(A + \sum_{p=1}^{i-1} e_p\bar{E}) + 2I = 0 \tag{12}$$

and using the condition (10).

Step 3: Check that the closed loop system (S_i) is stable. If this is the case, return to Step 2. If not,

$$\bar{e} = \sum_{p=0}^{i-1} e_p \tag{13}$$

Step 4: The largest $\bar{e}$ is defined as

$$\bar{e} = \sum_{p=0}^{\infty} e_p \tag{14}$$

Numerical Examples

Example 1

The normal, time-invariant stable matrix is

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$$

Table 1 Comparison of stability robustness results for structural perturbations

Perturbed elements of A	$a_{11}a_{12}$ $a_{21}a_{22}$	a_{11}	a_{12}	a_{21}	a_{22}	$a_{11}a_{12}$	$a_{11}a_{22}$	$a_{11}a_{21}$
U	1 1 1 1	1 0 0 0	0 1 0 0	0 0 1 0	0 0 0 1	1 1 0 0	1 0 0 1	1 0 1 0
$\mu_y = e$	0.2361	1.6569	1.6569	0.6558	0.3961	1.0000	0.3820	0.4805
μ_Q	0.3295	3.0000	2.0000	1.0000	0.6667	1.5201	0.5612	0.9150
$\bar{e}$	0.9913	2.9998	2.0000	1.5583	0.6660	1.9961	0.8981	2.9981

Table 1 gives bounds of allowable perturbations applying the criteria of Eqs. (3), (7), and (13) when the possible perturbed elements of E have different combinations. As can be seen, the new bounds $\bar{e}$ are a significant improvement over the bounds based on the criteria (3) and (7).

Example 2

We now consider the same example as the one considered in Refs. 11 and 15. The system chosen is the flare control of the Augmentor Wing Jet STOL Research Aircraft (AWJSRA). The equations for the longitudinal dynamics of the (AWJSRA) at an airspeed of 110 ft/s and flight path angle of -1 deg are given by

$$\dot{x}(t) = Ax + Bu \tag{15}$$

where

$$A = \begin{bmatrix} -0.0547 & -0.298 & -0.2639 & -0.0031 & 0.0 \\ 0.16 & -0.4712 & 0.4661 & 0.0437 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.1752 & 0.1236 & -0.1236 & -1.3 & 0.0 \\ -0.0174 & 1.92 & 0.0 & 0.0 & 0.0 \end{bmatrix} \tag{16}$$

$$B = \begin{bmatrix} -0.00315 & -0.0943 \\ 0.0408 & 0.0224 \\ 0.0 & 0.0 \\ -1.1200 & -0.08 \\ 0.0 & 0.0 \end{bmatrix} \tag{17}$$

The performance index considered is

$$J = \int_0^\infty (x^TQx + u^TRu)\mathrm{d}t \tag{18}$$

with $R = \mathrm{diag}\ [16, 0.5]$ and $Q = qI_5$.

In Ref. 11, the parameter q was used as a design parameter in order to improve the stability robustness bounds of the nominal closed loop system.

Table 2 Variations of μ_y and $\bar{e}$ with q, for LQ design

q	μ_y	$\bar{e}$	No. of iterations
0.1	0.1218900×10^{-2}	0.1560758×10^{-1}	72
0.25	0.1865912×10^{-2}	0.2197875×10^{-1}	67
0.5	0.2505332×10^{-2}	0.2816889×10^{-1}	64
1.0	0.3274987×10^{-2}	0.3601452×10^{-1}	64
5.0	0.5454320×10^{-2}	0.6323033×10^{-1}	72
10.0	0.6445775×10^{-2}	0.7935480×10^{-1}	76
50.0	0.8408343×10^{-2}	0.1238874	84
10^2	0.9019857×10^{-2}	0.1434407	86
10^4	0.1059858×10^{-1}	0.2189010	90

Table 3 Variations of μ_y and $\bar{e}$ with $\bar{q}$, for output feedback design

q	μ_y	$\bar{e}$	No of iterations
0.1	0.5642174×10^{-3}	0.1907418×10^{-1}	72
0.25	0.7541758×10^{-3}	0.2281746×10^{-1}	67
0.5	0.8740879×10^{-3}	0.2597963×10^{-1}	64
1.0	0.9119264×10^{-3}	0.2784040×10^{-1}	64
5.0	0.3168635×10^{-3}	0.9833818×10^{-2}	72
7.0	0.4861198×10^{-6}	0.1729670×10^{-4}	199

The stability robustness bounds μ_y and $\bar{e}$ and their variations with q are summarized in Table 2. Clearly it is seen that $\bar{e}$ is much greater than μ_y for the values of q considered. Now, a designer is able to define and determine precisely the allowable perturbations that do not disturb system stability. In this way, he can select an appropriate value for q that leads to satisfactory stability robustness bounds.

It should be pointed out that a greater value of the weighting parameter q leads to a more robust design. However, this conclusion is correct only for the full-state feedback based on LQ design methodology. In many practical applications, the whole-state vector is rarely available for measurement. A realistic control scheme in such cases involves the use of incomplete state measurement to form a control feedback.

In this case, the output feedback design was based on the output matrix

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{19}$$

Table 3 gives the stability robustness bounds μ_y and $\bar{e}$ and their variations with q. Clearly, it is seen that $\bar{e}$ is much greater than μ_y for the values of q considered. Table 3 also shows that $q = 1$ attains the most robust design. Therefore, in the case of output feedback, the greater value for q does not automatically mean a more robust design. The presented results can be used to select an appropriate value of the weighting parameter q to attain a robust design.

Finally, it should be pointed out that, in the case of output feedback control, the closed loop system might become unstable for large values of the weighting parameter q. Table 4 gives the closed loop poles of the output feedback system for different values of the parameter q. It is seen that for $q = 8$ the system becomes unstable.

Conclusion

A new computationally efficient algorithm has been proposed for stability robustness evaluation of linear time-invariant systems in state space models. Unlike many other criteria, this algorithm allows a designer to easily incorporate the directional information on structural perturbations in stability robustness analysis. In this way, the stability robustness bounds have been increased substantially over the ones recently reported in the literature. An aircraft control example has been given to illustrate the algorithm.

Acknowledgment

This research was supported in part by the U.S./Yugoslav Joint Fund for Scientific and Technological Cooperation in cooperation with the U.S. Department of Energy under Grant PP-727.

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Table 4 Closed-loop poles for different values of q

	$q = 7$	$q = 7.5$	$q = 8$
λ	-2.705	-2.739	-2.770
$\lambda_{2/3}$	$-0.1943 \times 10^{-2} \pm j0.6228$	$-0.1622 \times 10^{-4} \pm j0.6289$	$+0.1787 \times 10^{-2} \pm j0.6347$
$\lambda_{4/5}$	$-0.4100 \pm j0.116$	$-0.4166 \pm j0.1240$	$-0.4229 \pm j0.1307$

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Optimal Evasion Against a Proportionally Guided Pursuer

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Introduction

PURSUIT-EVASION problems traditionally have been classified among the classical examples of differential-game theory.[1] In recent years a different approach[2-5] has been applied to these problems, namely, to fix the pursuer's strategy and to form a one-sided optimal control problem for the evader. This approach, being conceptually simpler than the former, enables more realistic models to be applied for the dynamics of the opponents. In general, the fixed pursuer's strategy has been taken as constant-gain proportional navigation, which, under some formulations, is an optimal strategy for the pursuer. The optimal control analyses have demonstrated, by applying a more realistic model for the pursuer's dynamics (such as pure time delay[3] or some finite-order system[2,4]), that the evader can guarantee a fine miss distance even in conflict with a pursuer of unlimited maneuverability. However, the formulations employed in Refs. 2–5 do not treat the evader with the same degree of accuracy as the pursuer. Evader dynamics are either ignored (assuming an ideal system) or approximated by imposing some bounds.

Our objective in this Note is to provide an analysis for the optimal evasion problem by applying equally realistic models for the pursuer and the evader, and to carry out a parametric study considering the more important parameters of these models and their influence on the optimal strategies and the payoffs. The model to be used is linear, because of the relative complexity of the problem. Consequently, the results will be valid in the vicinity of the nominal collision courses.

Mathematical Modeling

We shall make the following assumptions:

1) The pursuit-evasion conflict is two-dimensional in the horizontal plane.

2) The speeds of the pursuer (P) and the evader (E) are constant.

3) The trajectories of P and E can be linearized around their collision triangle.

4) P applies a fixed-gain proportional navigation.

5) E has complete information on P's system and on the collision course.

6) Each vehicle's acceleration is subject to a first-order lag.

7) E's lateral acceleration is bounded. (P's lateral acceleration may or may not be bounded.)

Referring to Fig. 1, by assumptions 1–3 above we get the following equations:

$$\sin(\gamma_{e_0} + \gamma_e) \doteq \sin(\gamma_{e_0}) + \cos(\gamma_{e_0})\gamma_e \tag{1}$$

$$\dot{R} = V_p \cos(\gamma_{p_0}) - V_e \cos(\gamma_{e_0}) \equiv V_p' - V_e' = \text{const} \tag{2}$$

and

$$\dot{y} = \dot{y}_e - \dot{y}_p = V_e \cos(\gamma_{e_0})\gamma_e - V_p \cos(\gamma_{p_0})\gamma_p \tag{3}$$

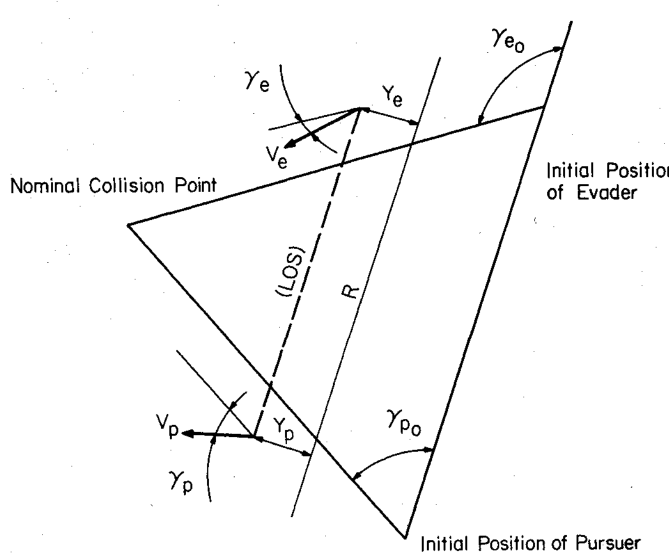

Fig. 1 Problem geometry.

Received Dec. 21, 1987; revision received April 17, 1988.

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